応用力学研究所研究集会報告 No.16ME-S1
「非線形波動の物理と数理構造」（研究代表者　梶原 健司）

Reports of RIAM Symposium No.16ME-S1
Physics and Mathematical Structures of Nonlinear Waves
Proceedings of a symposium held at Chikushi Campus, Kyushu Universiy,
Kasuga, Fukuoka, Japan, November 15 - 17, 2004

Article No. 24

The Dirichlet and the Poncelet problems

V. P. Burskii，A. S. Zhedanov

(Received March 21, 2005)

Research Institute for Applied Mechanics
Kyushu University
April, 2005

THE DIRICHLET AND THE PONCELET PROBLEMS

V.P. BURSKII AND A.S. ZHEDANOV

This work is devoted to a connection between ill-posed boundary value problems in a bounded semialgebraic domain for partial differential equations and the Poncelet problem, recently revealed by authors. The Poncelet problem is one of famous problems of projective geometry and it by itself has numerous links with a set of different problems of analysis and physics [2], [5], [10]. Investigations of ill-posed boundary value problems in bounded domains for partial differential equations go back to J.Hadamard [6]. Then A.Huber noticed [7] nonuniqueness of the solution of the Dirichlet problem for the equation of string vibration (string equation) in a rectangular (see reviews and results for more general domains in [9]). In [3] D.Burgin and R.Duffin have examined the homogeneous Dirichlet problem for the equation $\Phi_{tt} - \Phi_{xx} = 0$ in a rectangular $\{0 \le t \le T; 0 \le x \le X\}$. It was shown that if the ratio T/X is irrational, uniqueness in space of continuously differentiated functions with summable second derivatives takes place. In [1] V.I.Arnold, applying the results on mappings of a circle onto self, showed similar statements for classical solutions of the Dirichlet problem with the string equation in an ellipse. For nonrectangular domains the Dirichlet problem for the string equation was studied in connection with Denjoy-Poincare rotation number of homeomorphism of domain boundary, constructed on characteristics of equation (so-called an automorphism of characteristic billiards) by Fritz John [8].

In this paper we will examine the Dirichlet problem for the string equation

$$\Phi_{uv} = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\Phi|_C = 0 \quad \text{on} \quad C = \partial\Omega, \tag{2}$$

in a bounded semialgebraic domain, the boundary of which is given by some biquadratic algebraic curve

$$F(u, v) := \sum_{i,k=0}^{2} a_{ik} u^i v^k = 0. \tag{3}$$

As far as we know, the John algorithm for such curve was first considered in [4], where it was shown that corresponding discrete dynamical system is completely integrable and some solutions in terms of elliptic functions were presented in special cases.

Recall that the John algorithm is defined as follows [8].

Let Ω be arbitrary bounded domain, which is convex with respect to characteristic directions, i.e. it has the boundary C intersected in at most two points by each direct line that is parallel to x- and y-axes. We start from arbitrary point M_1 on C and consider a vertical line passing through M_1. Obviously, there exactly two points of intersection with the curve C: M_1 and some M_2. We denote I_1 an involution which transform M_1 into M_2. Then, starting from M_2, we consider a horizontal line passing through M_2. Let M_3 be the second point of intersection with the curve C. Let I_2 be corresponding involution: $I_2M_2 = M_3$. We then repeat this process, applying step-by-step involutions I_1 and I_2. Denote $T = I_2I_1$, $T^{-1} = I_1I_2$. This transformation $T : C \to C$ produces a discrete dynamical system on C, i.e. an action of group $\mathbb{Z}$ and each point $M \in C$ generates an orbit $\{T^nM|n \in \mathbb{Z}\}$. This orbit can be either finite or infinite. The point M corresponding to a finite orbit is called a *periodic* point and minimal n, for which $T^nM = M$, is called *a period* of the point M. F.John have proved several usefull assertions, among of which we extraxt the following one.

Sufficient condition of uniqueness. The homogeneous Dirichlet problem has only trivial solution if there are no periodic points on C.

For the case when there are periodic points the situation may be more complicated and an additional analysis is needed.

Our crucial observation is the following. It appears that the John algorithm for the bi-quadratic curve C is equivalent to so-called Poncelet algorithm for two arbitrary conics in two-dimensional affine space. Recall [2] that the Poncelet algorithm can be formulated as follows. Let A and B be two arbitrary conics (say, two ellipsis, this cases is general and can be achieved by an appropriate projective transformation). We start from an arbitrary point M_1 on a conic B and pass a tangent to a conic A. This tangent intersects the conic B in another point M_2. Then we pass another tangent from the point M_2 to a conic A. We obtain then the point M_3 on the conic B. This process can be repeated generating a set of the points $M_1, M_2, \dots M_n, \dots$ on the conic B. We denote also $L_1, L_2, \dots$ corresponding tangential points on the conic A. The famous big Poncelet theorem [2] states that if this algorithm is periodic for M_1, (i.e. $M_1 = M_N$ for some $N = 3, 4, \dots$) then this property doesn't depend from the choice of initial point M_1 on the conic B. Obviously, the same property is valid for the tangent points: $L_1 = L_N$. Thus the periodicity property for the Poncelet problem depends only on a choice of two conics A and B and do not depend on choice of initial point.

It is well known [2] that every affine conic admits a rational parametrization. For the conic A we can present this parametrization in the form:

$$x(u) = E_1(u)/E_0(u), \; y(u) = E_2(u)/E_0(u), \tag{4}$$

where $E_0(u), E_1(u), E_2(u)$ are some polynomials in u of degrees not exceeding 2. Analogously, for the conic B we have the parametrization

$$x(v) = G_1(v)/G_0(v), \; y(v) = G_2(v)/G_0(v), \tag{5}$$

where $G_0(v), G_1(v), G_2(v)$ are corresponding polynomials in v.

Our main result is the following

Theorem 1. Assume that the conics A and B are parametrized by (4), (5). Then the points $L_n = u_n$ and $M_n = v_n$ of the Poncelet problem are obtained by the John algorithm for some bi-quadratic curve $F(u, v) = 0$. Conversely, assume that a generic bi-quadratic curve $F(u, v) = 0$ is given. Then the John algorithm for this curve is equivalent to the Poncelet algorithm for two appropriately chosen conics A and B. These conics are not unique: there is an infinite family of such conics corresponding to the same bi-quadratic curve $F(u, v) = 0$. However all these families are projectively equivalent. The periodicity condition for the Poncelet problem is equivalent to periodicity condition for the John algorithm.

This theorem allows one to establish some important properties of the John algorithm for generic non-degenerated bi-quadratic curve $F(u, v) = 0$. For example, the Poncelet theorem says then that the periodicity property of the John algorithm depends only on equation of the bi-quadratic curve $F(u,v)=0$ and does not depend on the choice of initial point on this curve. Thus the bi-quadratic curve is *transitive* with respect to the John algorithm. This means that only one of two possibilities occurs:

(i) either all points of the curve C are non-periodic;

(ii) or all points of the bi-quadratic curve have the same finite period N.

Moreover, we can establish an explicit criterion for periodicity condition for the John algorithm. This formula comes from the famous Cayley criterion for the Poncelet problem. Recall that the Cayley criterion can be formulated as follows [2]. Let $f(\lambda)=det(A-\lambda B)$ is a characteristic determinant for the one-parameter pencil of conics A and B presented in the projective form. In more details, assume that the conic A has an affine equation $\phi_A(x,y)=0$. We then pass to projective co-ordinates $x=\xi_1/\xi_0, y=\xi_2/\xi_0$ and present the equation of the conic A in the form

$$\sum_{i,k=0}^{2} A_{ik}\xi_i\xi_k=0$$

with some 3×3 matrix A. Analogously we present projective equation for the conic B in the form

$$\sum_{i,k=0}^{2} B_{ik}\xi_i\xi_k=0$$

with some 3×3 matrix B. Then we define the function $f(\lambda)=det(A-\lambda B)$ constructed from these matrices A,B. The next step is construction of the Taylor series

$$\sqrt{f(\lambda)}=c_0+c_1\lambda+\ldots c_n\lambda^n+\ldots \tag{6}$$

Then we compute the Hankel-type determinants:

$$H_p^{(1)}=\begin{vmatrix} c_3 & c_4 & \cdots & c_{p+1}\\ c_4 & c_5 & \cdots & c_{p+2}\\ \cdots & \cdots & \cdots & \cdots\\ c_{p+1} & c_{p+2} & \cdots & c_{2p-1}\end{vmatrix},\quad p=2,3,4,\ldots \tag{7}$$

and

$$H_p^{(2)}=\begin{vmatrix} c_2 & c_3 & \cdots & c_{p+1}\\ c_3 & c_4 & \cdots & c_{p+2}\\ \cdots & \cdots & \cdots & \cdots\\ c_{p+1} & c_{p+2} & \cdots & c_{2p}\end{vmatrix},\quad p=1,2,3,\ldots \tag{8}$$

Then the Cayley criterion [2], [5] is: the trajectory of the Poncelet problem is periodic with the period N if and only if $H_p^{(1)}=0$, if $N=2p$ and $H_p^{(2)}=0$, if $N=2p+1$. For modern proof of the Cayley criterion see, e.g. [5].

Thus, in order to decide whether or not the John algorithm for the given bi-quadratic curve $F(u,v)=0$ is periodic we first pass from the John algorithm to the corresponding Poncelet problem and then apply the Cayley criterion.

Moreover, we give explicit solution of the John algorithm for a non-degenerated bi-quadratic curve $F(u,v)=0$:

$$u_n=\phi(q(n-n_1)),\ v_n=\psi(q(n-n_2)) \tag{9}$$

where $\phi(z), \psi(z)$ are two different elliptic functions of the second order with the same periods; n_1, n_2 are some parameters depending on initial conditions. The parameter q and the periods $2\omega_1, 2\omega_2$ of the elliptic functions $\phi(z), \psi(z)$ do not depend on initial conditions. Recall that an elliptic function of the second order $\phi(z)$ is characterized by the property that it has exactly two zeroes and two poles in the fundamental parallelogram of periods. It is well known [11] that generic elliptic function of the second order can be presented in one of two equivalent forms: either

$$\phi(z) = \kappa\, (a\wp(z) + b)/(c\wp(z) + d)$$

or

$$\phi(z) = \rho \frac{\sigma(z - e_1)\sigma(z - e_2)}{\sigma(z - d_1)\sigma(z - d_2)},$$

where $\wp(z)$ and $\sigma(z)$ are standard Weierstrass functions. There is the only restriction $d_1 + d_2 = e_1 + e_2$ for the parameters.

Thus we can write down general solution of the John algorithm in the form

$$u_n = \kappa_1 \frac{a_1\wp(q(n - n_1) + b_1}{c_1\wp(q(n - n_1) + d_1}, \quad v_n = \kappa_2 \frac{a_2\wp(q(n - n_2) + b_2}{c_2\wp(q(n - n_2) + d_2}$$

with some parameters $\kappa_i, a_i, b_i, c_i, d_i, n_i,\ i = 1, 2$.

Periodicity condition for the John algorithm is

$$qN = 2\omega_1 m_1 + 2\omega_2 m_2, \tag{10}$$

where N, m_1, m_2 are integers.

Finally, we announce a new result concerning solutions for the Dirichlet problem for the string equation in case of periodicity of the John algorithm.

Theorem 2. Let C be a nondegenerate bi-quadratic curve (3). The homogeneous Dirichlet problem (1),(2) has a nontrivial solution $\Phi \in C^2(\Omega)$ iff the John algorithm is periodical, i.e. the condition (10) is fufilled. In this case there is an infinite set of linear independent rational solutions.

Assume that the John algorithm for the curve (3) is periodic with period $N \geq 3$. Then it appears that the Dirichlet problem for the string equation is not unique and for every period $N \geq 3$ there exists an infinite series of rational functions $R_n(z; N), T_n(z; N)$ such that the string equation $\Phi_{uv} = 0$ has non-zero solutions $\Phi(u, v) = R_n(u; N) - R_n(v; N),\ n = 1, 2, \dots$ inside of bi-quadratic curve C. The rational functions $R_n(z; N), T_n(z; N)$ possess the property $R_n(u; N) = R_n(v; N)$ on the curve C providing that boundary condition $\Phi(u, v)\,|_C = 0$ holds on the curve C.

The concrete form of the rational functions $R_n(z; N), T_n(z; N)$ is connected with multiplication theorems for elliptic functions. Details of this connection will be published elsewhere.

References

[1] V. I. Arnold, *Small demominators. I*,– Izvestija AN SSSR, serija matematicheskaja, 25(1961), **1**, p.21-86.

[2] M. Berger, *Géométrie*, CEDIC, Paris, 1978.

[3] D.G.Bourgin and R. Duffin, *The Dirichlet problem for the vibrating string equation*, Bull.Am.Math.Soc., **45** (1939), 851–859.

[4] J.P.Francoise and O.Ragnisco, *An iterative process on quartics and integrable symplectic maps*, in "Symmetries and integrability of difference equations", P.A.Clarkson and F.W.Nijhoff eds., Cambridge University Press 1998.

[5] P. Griffiths and J. Harris, *On a Cayley's explicit solution to Poncelet's porism*, Enseign. Math. (2), **24** (1978), 31–40.
[6] J. Hadamard, Equations aux derivees partielles, L'Enseignment Mathematique, Vol. 36(1936), p. 25-42.
[7] A. Huber, *Erste Randwertaufgabe fur geschlossene Bereiche bei der Gleichung* $U_{xy} = f(x, y)$, Monatshefte für Mathematik und Physik, 1932, 39, p. 79-100.
[8] F.John, *The Dirichlet problem for a hyperbolic equation*, Am.J.Math. **63** (1941), 141–154.
[9] B. Yo. Ptashnik, *Incorrect boundary value problems for differential equations with partual derivatives*, Kiev, Naukova dumka, 1984.(In Russian)
[10] A.P. Veselov, *Integrable systems with discrete time and difference operators*, Functional Analysis and its Applications **22** (1988), 1–13 (Russian).
[11] E.T. Whittacker, G.N. Watson, *A Course of Modern Analysis*, Cambridge, 1927.

Institute of Applied Mathematics, Donetsk, 83114, Ukraine

Donetsk Institute for Physics and Technology, Donetsk 83114, Ukraine