Probability distribution of estimation error on current vector measurement using multiple HF ocean surface radars

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Estimation of velocity vector

- Single HFOSR: Radial component only
- Two HFOSR: 2D-velocity vector
  - Directory influenced of measurement error
- Three or more HFOSR: 2D-velocity vector
  - Able to reduce the influence of measurement error by statistical method
    - Measurement error of each HFOSR
    - Influence of wind direction on SNR
    - Important to build large scale observing system using HFOSR

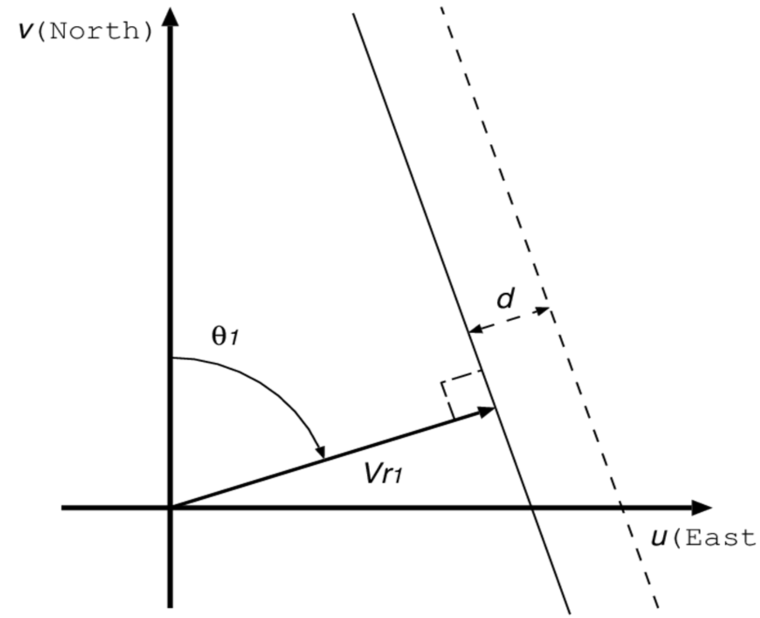
Assumption

- Each measurement includes measurement error

- Assumption of probability density function for each measurement
  - 2D-Gaussian distribution with average and variance
  - Probability density function of 2D-velocity vector
    → product of probability density function for each measurement
    - Capable to use multi HFOSRs to estimate 2D-velocity vector
    - (Taking into consideration different RMS error of each HFOSR)

Probability density function for radial velocity

- Probability density of current vector is a function of the square of distance to the perpendicular line to the radial velocity vector
- Azimuth of line of sight θ_i
- Measured radial velocity V_i
- Root Mean Square Error σ



$$p_i'(d_i) = \frac{1}{\sqrt{2\pi}\sigma} \int \exp\{-\frac{d_i^2}{2\sigma^2}\}$$

$$d_i^2(u, v) = (u \sin\theta_i + v \cos\theta_i - V_i)^2$$

$$p_i'(u, v) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(u \sin\theta_i + v \cos\theta_i - V_i)^2}{2\sigma^2}\}$$

Probability density function of current vector

- Probability density function (pdf) of current vector is the normalized product of pdfs for each radial velocity.

$$P(u,v) \propto \prod_{i=1}^{N} \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(u\sin\theta_i + v\cos\theta_i - V_i)^2}{2\sigma^2}\} \right)$$

$$= \frac{P'(u,v)}{\iint P'(u,v)dudv}$$

$$P'(u,v) = \prod_{i=1}^{N} \exp\{-\frac{(u\sin\theta_i + v\cos\theta_i - V_i)^2}{2\sigma^2}\}$$

$$= \exp\{-\frac{E^2(u,v)}{2\sigma^2}\}$$

Probability density function of current vector

$$
\begin{aligned}
E^2(u,v) &= \sum_{i=1}^{N}(u\sin\theta_i + v\cos\theta_i - V_i)^2 \\
&= u^2\sum_{i=1}^{N}\sin^2\theta_i + 2uv\sum_{i=1}^{N}\sin\theta_i\cos\theta_i + v^2\sum_{i=1}^{N}\cos^2\theta_i \\
&\quad - 2u\sum_{i=1}^{N}V_i\sin\theta_i - 2v\sum_{i=1}^{N}V_i\cos\theta_i + \sum_{i=1}^{N}V_i^2 \\
&= u^2A + 2uvH + v^2B - 2uG - 2vF + C
\end{aligned}
$$

Probability density function of current vector

$$E^2(u,v) = u^2 A + 2uvH + v^2 B - 2uG - 2vF + C$$

$$D = AB - H^2$$

$$= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sin^2(\theta_i - \theta_j) \geq 0$$

$D = 0$: Two parallel lines or Parabola $\iint P'(u,v)dudv = \infty$

$D \neq 0$: Ellipse $\iint P'(u,v)dudv = \dfrac{2\pi\sigma^2}{\sqrt{D}}$

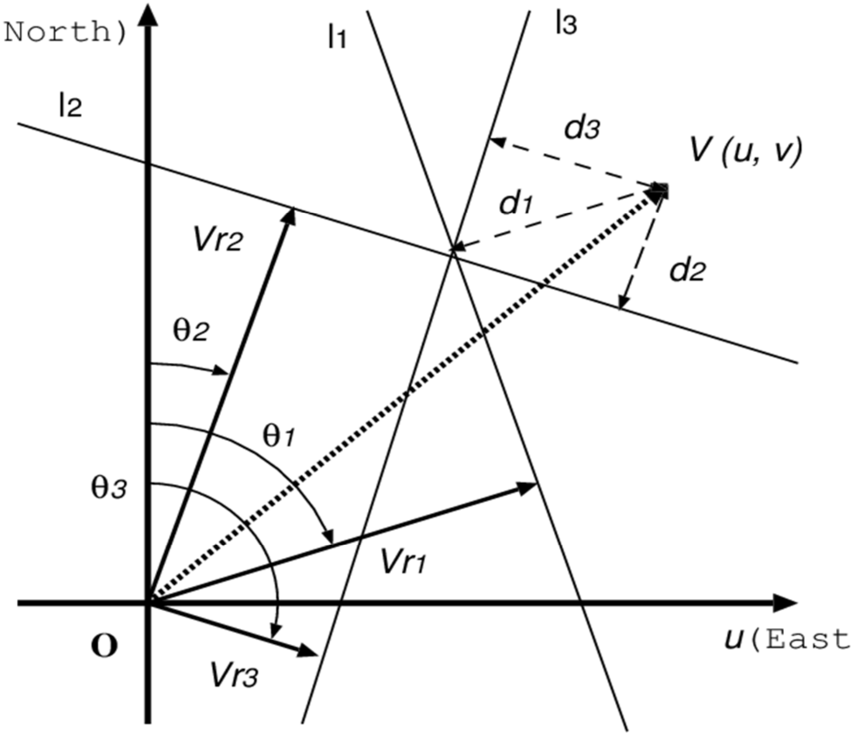
$$P(u,v) = \frac{\sqrt{D}}{2\pi\sigma^2} \exp\{-\frac{E^2(u,v)}{2\sigma^2}\}$$

Probability density function for 2 and 3 HFOSR observation

$$P_3(u,v) = \frac{\sqrt{D_3}}{2\pi\sigma^2}\exp\{-\frac{E_3^2(u,v)}{2\sigma^2}\}$$

$$D_3 = \sin^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_3) + \sin^2(\theta_2-\theta_3)$$

$$E_3^2 = u^2(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3) + 2uv(\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2 + \sin\theta_3\cos\theta_3) + v^2(\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3)$$
$$- 2u(V_1\sin\theta_1 + V_2\sin\theta_2 + V_3\sin\theta_3) - 2v(V_1\cos\theta_1 + V_2\cos\theta_2 + V_3\cos\theta_3) + (V_1^2 + V_2^2 + V_3^2)$$



$$P_2(u,v) = \frac{\sqrt{D_2}}{2\pi\sigma^2}\exp\{-\frac{E_2^2(u,v)}{2\sigma^2}\}$$

$$D_2 = \sin^2(\theta_1-\theta_2)$$

$$E_2^2 = u^2(\sin^2\theta_1 + \sin^2\theta_2) + 2uv(\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2) + v^2(\cos^2\theta_1 + \cos^2\theta_2)$$
$$- 2u(V_1\sin\theta_1 + V_2\sin\theta_2) - 2v(V_1\cos\theta_1 + V_2\cos\theta_2) + (V_1^2 + V_2^2)$$

Estimated current vector

1st moments of pdf

$$u_e = \iint uP(u,v)dudv = \frac{BG - HF}{D}$$

$$= \frac{1}{D}\{(\sum_{i=1}^{N}\cos^2\theta_i)(\sum_{i=1}^{N}V_i\sin\theta_i) - (\sum_{i=1}^{N}\sin\theta_i\cos\theta_i)(\sum_{i=1}^{N}V_i\cos\theta_i)\}$$

$$v_e = \iint vP(u,v)dudv = \frac{AF - HG}{D}$$

$$= \frac{1}{D}\{(\sum_{i=1}^{N}\sin^2\theta_i)(\sum_{i=1}^{N}V_i\cos\theta_i) - (\sum_{i=1}^{N}\sin\theta_i\cos\theta_i)(\sum_{i=1}^{N}V_i\sin\theta_i)\}$$

Variance of estimated current vector

2nd moments of pdf

$$\sigma_u^2 = \iint (u - u_e)^2 P(u,v)dudv = \frac{B}{D} = \frac{\sigma^2}{D}\sum_{i=1}^{N}\cos^2\theta_i$$

$$\sigma_v^2 = \iint (v - v_e)^2 P(u,v)dudv = \frac{A}{D} = \frac{\sigma^2}{D}\sum_{i=1}^{N}\sin^2\theta_i$$

$$\sigma_{abs}^2 = \iint \{(u - u_e)^2 + (v - v_e)^2\} P(u,v)dudv = \frac{A+B}{D} = \frac{\sigma^2}{D}N$$

Shape of iso-probability ellipse

$$P(u,v) = \frac{\sqrt{D}}{2\pi\sigma^2}\exp\{-\frac{E^2(u,v)}{2\sigma^2}\}$$

$$E^2(u,v) = u^2A + 2uvH + v^2B - 2uG - 2vF + C$$

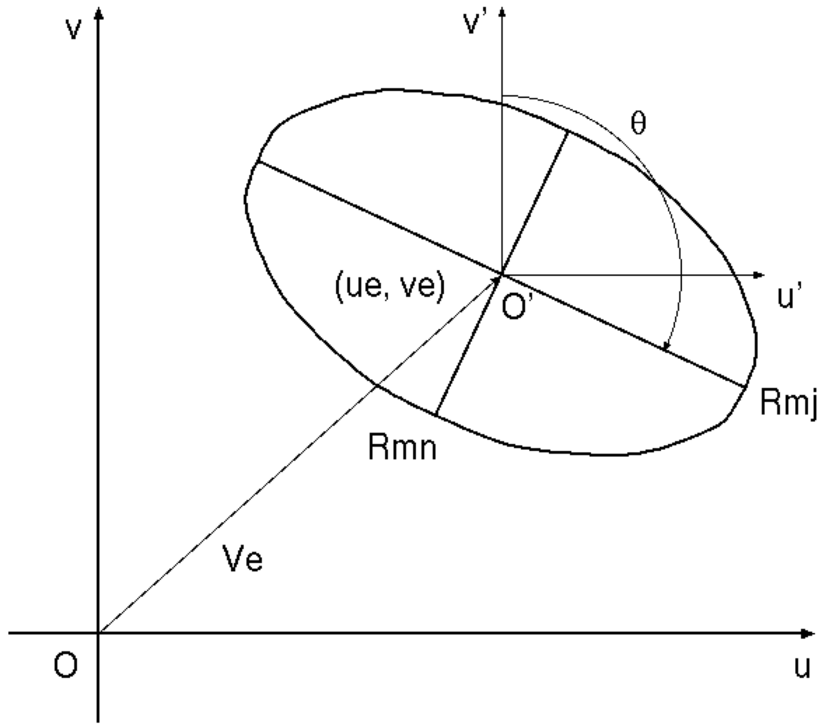
Parallel translation $(u', v') = (u + u_e, v + v_e)$

$$P(u',v') = \frac{\sqrt{D}}{2\pi\sigma^2}\exp\{-\frac{u'^2A + 2u'v'H + v'^2B}{2\sigma^2}\}$$

Iso-probability ellipse with probability *p*

$$u'^2A + 2u'v'H + v'^2B = -2\sigma^2 \ln\frac{2\pi\sigma^2 p}{\sqrt{D}} = \sigma^2 p'$$

Shape of iso-probability ellipse



(u_e, V_e): Estimated current vector

u', v': Parallel translated axes

Θ: Azimuth of major axis

R_{mj}: Radius of major axis

R_{mn}: Radius of minor axis

Radii of iso-probability ellipse

$H = 0$ and $A = B$:

iso-probability ellipse is a circle

$$R = \sigma\sqrt{\frac{p'}{A}}$$

Radii of iso-probability ellipse

Otherwise,

$$R_{mj} = \sigma\sqrt{\frac{p'}{2D}}\sqrt{(A+B)+\sqrt{(A-B)^2+4H^2}}$$

$$= \sigma\sqrt{\frac{p'}{2D}}\sqrt{N+\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\theta_i-\theta_j)}}$$

$$R_{mn} = \sigma\sqrt{\frac{p'}{2D}}\sqrt{(A+B)-\sqrt{(A-B)^2+4H^2}}$$

$$= \sigma\sqrt{\frac{p'}{2D}}\sqrt{N-\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\theta_i-\theta_j)}}$$

Azimuth of major axis

$H = 0$: Major axis agrees with u- or v-axis.

$$A > B \quad \theta_{mj} = 0$$

$$A < B \quad \theta_{mj} = \pi/2$$

Otherwise,

$$\theta_{mj} = \begin{cases} \theta' & (\theta' > 0, H > 0) \\ \theta' + \pi/2 & (\theta' < 0, H > 0) \\ \theta' - \pi/2 & (\theta' > 0, H < 0) \\ \theta' & (\theta' < 0, H < 0) \end{cases}$$

$$\theta' = \frac{1}{2} \arctan \frac{2H}{B - A}$$

$$= \frac{1}{2} \arctan \frac{\sum_{i=1}^{N} \sin 2\theta_i}{\sum_{i=1}^{N} \cos 2\theta_i}$$

Valiance along major and minor axes

2nd moment of pdf along the major and minor axes

$$\sigma_{mj}^2 = \frac{\sigma^2}{D}\{(A+B) + \sqrt{(A-B)^2 + 4H^2}\}$$

$$= \frac{\sigma^2}{D}\{N + \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\theta_i - \theta_j)}\}$$

$$\sigma_{mn}^2 = \frac{\sigma^2}{D}\{(A+B) - \sqrt{(A-B)^2 + 4H^2}\}$$

$$= \frac{\sigma^2}{D}\{N - \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\theta_i - \theta_j)}\}$$

Summary

- By assuming the probability density function of radial velocity, the probability density distribution of 2D-velocity vector is calculated.
- Using the probability distribution function, the most reliable 2D-velocity vector is estimated.
- Spatial probability distribution of error vector is analyzed.
- To evaluate the measurement error, the probability density distribution of 2D-velocity vector is important.
- These results are quite useful to build the large scale observing system using HFOSR based on many independent HFOSR systems.